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LETTER TO THE EDITOR

Experimental tests of quantum mechanics versus local hidden variable theories

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Abstract. We propose new inequalities involving polarisation correlation parameters as tests of local hidden variable theories versus quantum mechanics. These are derived using Bell's formulation of Einstein's locality condition.

1. Introduction

Einstein *et al* (1935) argued that quantum mechanics cannot be a complete theory by means of a discussion of measurements on two spatially separated systems S_1 and S_2 which have interacted in the past. Their essential assumption is the locality condition of Einstein (1949): 'The real factual situation of the system S_2 is independent of what is done with the system S_1, which is spatially separated from the former.' From this locality condition, Bell (1964, 1971) and Clauser *et al* (1969) derived an important inequality which is the basis of Bell's theorem that no local (deterministic or stochastic) hidden variable theory can reproduce all the experimental predictions of quantum mechanics. Experimental tests of quantum mechanics versus Bell's inequality are of fundamental importance (Freedman and Clauser 1972, Holt 1973, Holt and Pipkin 1974, Faraci *et al* 1974, Clauser 1976, Fry and Thompson 1976, Bruno *et al* 1977). New and refined experiments in this direction are in progress (Aspect 1975, Aspect and Imbert 1976). We propose new tests between quantum mechanics and theories obeying Einstein's locality condition as formulated by Bell, i.e. local hidden variable theories. It will be clear that, as in the case of Bell's inequality, the inequalities presented here can also be derived from other formulations (Clauser and Horne 1974, Bell 1975, Stapp 1976, d'Espagnat 1975, 1977, Eberhard 1977) of the locality condition. We state our results only for a system of two spin-$\frac{1}{2}$ particles; generalisation to other systems is straightforward using the arguments of Clauser *et al* (1969).

Consider a system of two spin-$\frac{1}{2}$ particles prepared in such a state that they move in different directions towards two measuring devices which measure spin components $A(=\pm 1)$ and $B(=\pm 1)$ along directions $\hat{a}$ and $\hat{b}$ respectively. Suppose that the initial state is described by hidden variables λ with probability distribution $\rho_0(\lambda)$. Bell (1971) characterises local hidden variable theories (deterministic or stochastic) as those in which the expectation values of A, B and AB in the state λ, denoted respectively by $\bar{A}$, $\bar{B}$ and $\overline{AB}$, obey the locality conditions,

$$\overline{AB}(\hat{a}, \hat{b}, \lambda) = \bar{A}(\hat{a}, \lambda)\bar{B}(\hat{b}, \lambda) \tag{1}$$

and the obvious inequalities,

$$[\bar{A}(\hat{a}, \lambda)]^2 \leqslant 1, \qquad [\bar{B}(\hat{b}, \lambda)]^2 \leqslant 1. \tag{2}$$

The chief point is that $\bar{A}(\bar{B})$ does not depend on the setting $\hat{b}(\hat{a})$ of the distant instrument. We further assume that the probability that both particles trigger the measuring devices depends only on λ, and denote it by $f(\lambda)$. Then, the mean value $P(\hat{a}, \hat{b})$ of the product AB is given by

$$P(\hat{a}, \hat{b}) = \int \mathrm{d}\lambda \rho(\lambda) \bar{A}(\hat{a}, \lambda) \bar{B}(\hat{b}, \lambda), \tag{3}$$

where

$$\rho(\lambda) \equiv \frac{\rho_0(\lambda) f(\lambda)}{\int \mathrm{d}\lambda' \rho_0(\lambda') f(\lambda')}.$$

Obviously

$$\int \mathrm{d}\lambda \rho(\lambda) = 1 \qquad \text{and} \qquad \rho(\lambda) \geqslant 0. \tag{4}$$

We shall obtain new consequences of this locality assumption which conflict with quantum mechanics and may be tested experimentally by measurement of the polarisation correlations $P(\hat{a}, \hat{b})$.

2. Method

All the new inequalities to be considered here, as well as those proposed by Bell earlier, involve polarisation correlation parameters $P(\hat{a}, \hat{b})$ linearly, i.e. they are of the form

$$J = \sum_{i=1}^{M} \sum_{j=1}^{N} C_{ij} P_{ij} \leqslant 1 \tag{5}$$

where $P_{ij} = P(\hat{a}_i, \hat{b}_j) = \int \mathrm{d}\lambda \rho(\lambda) x_i(\lambda) y_j(\lambda)$, $\bar{A}(\hat{a}_i, \lambda) \equiv x_i(\lambda)$, $\bar{B}(\hat{b}_j, \lambda) \equiv y_j(\lambda)$ and C_{ij}'s are constants independent of $\hat{a}_i$, $\hat{b}_j$. We now note: (i) As the form J depends on $x_i(\lambda)$ and $y_j(\lambda)$ linearly its maxima and minima will clearly be achieved on the boundary. We then need consider only the case $[x_i(\lambda)]^2 = [y_j(\lambda)]^2 = 1$ for finding the maxima of J under the conditions (2). (ii) The locality conditions (2) and (4) used to derive the inequalities (5) are invariant under

$$x_i(\lambda) \to x_i'(\lambda) = \xi_i x_i(\lambda), \qquad y_j(\lambda) \to y_j'(\lambda) = \eta_j y_j(\lambda)$$

where $(\xi_i)^2 = (\eta_j)^2 = 1$. Hence for every inequality of the form (5), the further inequalities obtained by replacing P_{ij} by $\xi_i \eta_j P_{ij}$,

$$\sum_{j=1}^{N} \sum_{i=1}^{M} \xi_i \eta_j C_{ij} P_{ij} \leqslant 1 \tag{5'}$$

must also hold.

The results we obtain are consequences of the basic inequality asserting that the square of the sum of an odd number of terms, each of which can take the value $+1$ or -1, is necessarily bounded below by 1, i.e.

$$[x_1(\lambda) + x_2(\lambda) + \ldots + x_n(\lambda) + y_1(\lambda) + \ldots + y_m(\lambda)]^2 \geqslant 1 \tag{6}$$

if $m+n=\text{odd}$ for $[x_i(\lambda)]^2=[y_j(\lambda)]^2=1$. The method is to form positive linear combinations of the inequalities of this type with different m and n values, such that the combination does not contain terms of the form $x_i(\lambda)x_j(\lambda)$ or $y_i(\lambda)y_j(\lambda)$ with $i\neq j$. These, on multiplying by $\rho(\lambda)$ and integrating over λ, and using (4), yield the desired inequalities of the type (5) and (5′).

In order to illustrate the above procedure let us first rederive Bell's inequality by our method. We have

$$[x_1(\lambda)-y_1(\lambda)-y_2(\lambda)]^2+[x_2(\lambda)-y_1(\lambda)+y_2(\lambda)]^2\geqslant 2$$

i.e.

$$x_1(\lambda)y_1(\lambda)+x_1(\lambda)y_2(\lambda)+x_2(\lambda)y_1(\lambda)-x_2(\lambda)y_2(\lambda)\leqslant 2$$

for $[x_i(\lambda)]^2=[y_i(\lambda)]^2=1$. On multiplying by $\rho(\lambda)$ and integrating over λ we get

$$\tfrac{1}{2}(P_{11}+P_{12}+P_{21}-P_{22})\leqslant 1.$$

Similarly,

$$\tfrac{1}{2}(\xi_1\eta_1P_{11}+\xi_1\eta_2P_{12}+\xi_2\eta_1P_{21}-\xi_2\eta_2P_{22})\leqslant 1 \tag{7}$$

for $\xi_i^2=\eta_j^2=1$; these are equivalent to Bell's inequalities.

3. New results

Let $\xi_i^2=\eta_j^2=1$, and

$$Q_{ij}\equiv\xi_i\eta_jP_{ij}. \tag{8}$$

Then we prove the following new inequalities:

$$Q_{11}+Q_{21}+Q_{31}+Q_{41}+Q_{12}+Q_{22}+Q_{32}-Q_{42}+Q_{13}-Q_{23}+Q_{24}-Q_{34}+Q_{15}-Q_{35}\leqslant 6. \tag{9}$$

$$Q_{11}+Q_{21}+Q_{31}+Q_{41}+Q_{12}-Q_{22}+Q_{13}-Q_{33}+Q_{14}-Q_{44}+Q_{25}-Q_{35}+Q_{26}-Q_{46}+Q_{37}-Q_{47}\leqslant 8. \tag{10}$$

$$\sum_{\substack{i=1,3,5,7\\ j=1,2,3}}(1-2\delta_{i,2j+1})(Q_{i,j}+Q_{i,j+3}+Q_{i+1,j}-Q_{i+1,j+3})\leqslant 16. \tag{11}$$

Further, the inequalities obtained from equations (9)–(11) by the interchange $Q_{ab}\leftrightarrow Q_{ba}$ are also valid; note that the inequalities so obtained are distinct from (9)–(11), because, in general $\hat{a}_i$ and $\hat{b}_i$ are unequal and hence Q_{ij} and Q_{ji} are also unequal.

Each of the above inequalities constitutes a large number of inequalities on the P_{ij} because of the freedom of choice of the ξ_i and η_i. As explained in § 2 it is sufficient to exhibit the proofs for the case $\xi_i=\eta_i=1$. Combining inequalities of the form (6) with $m+n=5$ and $m+n=3$ we obtain,

$$(x_1+x_2+x_3-y_1-y_2)^2+(x_4-y_1+y_2)^2+(x_1-x_2-y_3)^2+(x_2-x_3-y_4)^2+(x_3-x_1+y_5)^2\geqslant 5, \tag{12}$$

suppressing the λ dependence of the $x_i(\lambda)$ and $y_i(\lambda)$. Multiplying by $\rho(\lambda)$ and integrating over λ we obtain the inequality (9). Starting from the inequality

$$(x_1+x_2+x_3+x_4-y_1)^2+(x_1-x_2-y_2)^2+(x_1-x_3-y_3)^2+(x_1-x_4-y_4)^2 \\ +(x_2-x_3-y_5)^2+(x_2-x_4-y_6)^2+(x_3-x_4-y_7)^2 \geqslant 7, \tag{13}$$

we obtain similarly the result (10). Starting from the inequality

$$(x_1+x_2-y_1-y_2-y_3)^2+(x_1-x_2-y_4-y_5-y_6)^2 \\ +(x_3+x_4+y_1-y_2-y_3)^2+(x_3-x_4+y_4-y_5-y_6)^2 \\ +(x_5+x_6-y_1+y_2-y_3)^2+(x_5-x_6-y_4+y_5-y_6)^2 \\ +(x_7+x_8-y_1-y_2+y_3)^2+(x_7-x_8-y_4-y_5+y_6)^2 \geqslant 8, \tag{14}$$

we obtain the inequality (11). Finally, starting from the inequalities obtained from (12)–(14) by interchanging x_i and y_i we derive inequalities obtained from (9)–(11) by the interchange $Q_{ab} \leftrightarrow Q_{ba}$. Inequalities on P_{ij} arising from inequalities of the form (6) with $m+n>5$ will not be discussed here.

4. Comparison with Bell's inequalities and with quantum mechanics

We prove by simple examples that the new inequalities (9)–(11) indeed provide restrictions on the P_{ij} not implied by Bell's inequalities (7). Suppose $P_{42}=P_{23}=P_{34}=P_{35}=0$, and the remaining P_{ij} occurring in equation (9) are equal to $\frac{2}{3}$; then all Bell's inequalities involving these P_{ij} are obeyed, but the inequality (9) is violated. The choice $P_{i+1,j+3}=0$ for $i=1,3,5,7$, $j=1,2,3$, and the remaining P_{ij} occurring in equation (11) equal to $\frac{2}{3}$ respects all Bell's inequalities, but violates the inequality (11). The choice $P_{11}=P_{21}=P_{31}=P_{41}=\frac{1}{3}$, $P_{12}=P_{13}=P_{14}=P_{25}=P_{26}=P_{37}=\frac{2}{3}$, and $P_{22}=P_{33}=P_{44}=P_{35}=P_{46}=P_{47}=-\frac{2}{3}$ respects the relevant Bell inequalities but violates inequality (10).

The new inequalities, like Bell's inequalities, are in conflict with quantum mechanics. For example, if we choose $\xi_i=\eta_i=+1$, and $\hat{b}_1=\hat{a}_5$, $\hat{b}_2=\hat{a}_4$, $\hat{b}_3=\hat{a}_2$, $\hat{b}_4=\hat{b}_5=\hat{a}_3$, then equation (9) yields

$$\sum_{\substack{i,j=1\\ i<j}}^{5} P(\hat{a}_i,\hat{a}_j) \leqslant 6+P(\hat{a}_2,\hat{a}_2)+2P(\hat{a}_3,\hat{a}_3)+P(\hat{a}_4,\hat{a}_4). \tag{15}$$

In the Bohm and Aharonov (1957) example of two spin-$\frac{1}{2}$ particles produced in the single state, the quantum mechanical result is

$$P(\hat{a}_i,\hat{a}_j)|_{\mathrm{QM}} = -\hat{a}_i \cdot \hat{a}_j. \tag{16}$$

Now, choose $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$ in the same plane with an angle of $2\pi/3$ between each pair of vectors, and $\hat{a}_4+\hat{a}_5=0$; substituting the quantum mechanical result the left-hand side of the inequality (15) becomes $\frac{5}{2}$, and the right-hand side becomes 2, in clear violation of the inequality. Hence it would be possible to distinguish experimentally between quantum mechanics and the locality predictions given by equations (7), (9), (10), and (11).

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